

Stable Matching with Uncertain Linear Preferences *

Haris Aziz¹², Péter Biró³, Serge Gaspers²¹, Ronald de Haan⁴, Nicholas Mattei¹², and Baharak Rastegari⁵

¹ Data61 (formerly: NICTA), CSIRO, Sydney, Australia.

`haris.aziz@data61.csiro.au`, `nicholas.mattei@data61.csiro.au`

² University of New South Wales, Sydney, Australia. `sergeg@cse.unsw.edu.au`

³ Hungarian Academy of Sciences, Institute of Economics. `peter.biro@krtk.mta.hu`

⁴ TU Wien, Vienna, Austria. `dehaan@ac.tuwien.ac.at`

⁵ School of Computing Science, University of Glasgow, Glasgow, UK.
`baharak.rastegari@glasgow.ac.uk`

Abstract. We consider the two-sided stable matching setting in which there may be uncertainty about the agents' preferences due to limited information or communication. We consider three models of uncertainty: (1) lottery model — in which for each agent, there is a probability distribution over linear preferences, (2) compact indifference model — for each agent, a weak preference order is specified and each linear order compatible with the weak order is equally likely and (3) joint probability model — there is a lottery over preference profiles. For each of the models, we study the computational complexity of computing the stability probability of a given matching as well as finding a matching with the highest probability of being stable. We also examine more restricted problems such as deciding whether a certainly stable matching exists. We find a rich complexity landscape for these problems, indicating that the form uncertainty takes is significant.

1 Introduction

We consider a *Stable Marriage problem (SM)* in which there is a set of men and a set of women. Each man has a linear order over the women, and each woman has a linear order over the men. For the purpose of this paper we assume that the preference lists are complete, i.e., each agent finds each member of the opposite side acceptable.⁶ In the stable marriage problem the goal is to compute a *stable matching*; a matching where no two agents prefer to be matched to each other rather than be matched to their current partners. Unlike most of

* A preliminary version of this paper has been accepted for publication in the proceedings of the 9th International Symposium on Algorithmic Game Theory (SAGT 2016).

⁶ We note that the complexity of all problems that we study are the same for complete and incomplete lists, where non-listed agents are deemed unacceptable—see Proposition 2 in Section 3.1.

the literature on stable matching problems [6, 11, 14], we assume that men and women may have uncertainty in their preferences which can be captured by various probabilistic uncertainty models. We focus on *linear models* in which each possible deterministic preference profile is a set of linear orders.

Uncertainty in preferences could arise for a number of reasons both practical or epistemological. For example, an agent could express a weak order because the agent did not invest enough time or effort to differentiate between potential matches and therefore one could assume that each linear extension of the weak order is equally likely; this maps to our *compact indifference model*. In many real applications the ties are broken randomly with lotteries, e.g., in the school choice programs in New York and Boston as well as in centralized college admissions in Ireland. However, a central planner may also choose a matching that is optimal in some sense, without breaking the ties in the preference list. For instance, in Scotland they used to compute the maximum size (weakly) stable matching to allocate residents to hospitals [11]. We argue that another natural solution could be the matching which has the highest probability of being stable after conducting a lottery. Alternatively, there may be a cost associated with eliciting preferences from the agents, so a central planner may want to only obtain and provide a recommendation based on a subset of the complete orders [2].

As another example, imagine a group of interns are admitted to a company and allocated to different projects based on their preferences and the preferences of the project leaders. Suppose that after three months the interns can switch projects if the project leaders agree; though the company would prefer not to have swaps if possible. However, both the interns and the project leaders can have better information about each other after the three months, and the assignment should also be stable with regard to the refined preferences. This example motivates our lottery and joint probability models. In the *lottery model*, the agents have *independent* probabilities over possible linear orders (e.g. each project leader has a probability distribution on possible refined rankings over the interns independently from each other). In the *joint probability model*, the probability distribution is over possible preference profiles and can thus accommodate the possibility that the preferences of the agents are refined in a correlated way (e.g. if an intern performs well in the first three months then she is likely to be highly ranked by all project leaders). Uncertainty in preferences has already been studied in voting [8] and for cooperative games [10]. Ehlers and Massó [3] considers many-to-one matching markets under a Bayesian setting. Similarly, in auction theory, it is standard to examine Bayesian settings in which there is a probability distribution over the types of agents.

To illustrate the problem we describe a simple example with four agents. We write $b \succ_a c$ to say that agent a prefers b to c and assume the lottery model.

Example 1. We have two men m_1 and m_2 and two women w_1 and w_2 . Each agent assigns a probability to each strict preference ordering as follows. (i) $p(w_1 \succ_{m_1} w_2) = 0.4$ and $p(w_2 \succ_{m_1} w_1) = 0.6$ (ii) $p(w_1 \succ_{m_2} w_2) = 0.0$ and $p(w_2 \succ_{m_2} w_1) = 1.0$ (iii) $p(m_1 \succ_{w_1} m_2) = 1.0$ and $p(m_2 \succ_{w_1} m_1) = 0.0$ (iv) $p(m_1 \succ_{w_2} m_2) = 0.8$ and $p(m_2 \succ_{w_2} m_1) = 0.2$. This setting admits two match-

ings that are stable with positive probability: $\mu_1 = \{(m_1, w_1), (m_2, w_2)\}$ and $\mu_2 = \{(m_1, w_2), (m_2, w_1)\}$. Notice that if each agent submits the preference list that s/he finds most likely to be true, then the setting admits a unique stable matching that is μ_2 . The probability of μ_2 being stable, however, is 0.48 whereas the probability of μ_1 being stable is 0.52.

1.1 Uncertainty Models

We consider three different uncertainty models:

- **Lottery Model:** For each agent, we are given a probability distribution over strict preference lists.
- **Compact Indifference Model:** Each agent reports a single weak preference list that allows for ties. Each complete linear order extension of this weak order is assumed to be equally likely.
- **Joint Probability Model:** A probability distribution over preference profiles is specified.

Note that for the Lottery Model and the Joint Probability Model the representation of the input preferences can be exponentially large. However, in settings where similar models of uncertainty are used, including resident matching [2] and voting [8], a limited amount of uncertainty (i.e. small supports) is commonly expected and observed in real world data. Consequently, we consider special cases when the uncertainty is bounded in certain natural ways including the existence of only a small number of uncertain preferences and/or uncertainty on only one side of the market.

Observe that the compact indifference model can be represented as a lottery model. This is a special case of the lottery model in which each agent expresses a weak order over the candidates (similar to the SMT setting [6, 11]). However, the lottery model representation can be exponentially larger than the compact indifference model; for an agent that is indifferent among n agents on the other side of the market, there are $n!$ possible linearly ordered preferences.

1.2 Computational Problems

Given a stable marriage setting where agents have uncertain preferences, various natural computational problems arise. Let *stability probability* denote the probability that a matching is stable. We then consider the following two natural problems for each of our uncertainty models.

- **MATCHINGWITHHIGHESTSTABILITYPROBABILITY:** Given uncertain preferences of the agents, compute a matching with the highest stability probability.
- **STABILITYPROBABILITY:** Given a matching and uncertain preferences of the agents, what is the stability probability of the matching?

Problems	Lottery Model	Compact Indifference	Joint Probability
STABILITYPROBABILITY	#P-complete in P for all three models if 1 side is certain	?	in P
ISSTABILITYPROBABILITYNON-ZERO	NP-complete	in P	in P
ISSTABILITYPROBABILITYONE	in P	in P	in P
EXISTSPOSSIBLYSTABLEMATCHING	in P	in P	in P
EXISTSCERTAINLYSTABLEMATCHING	in P	in P	NP-complete
	?	NP-hard	NP-hard
MATCHINGWITHHIGHESTSTABILITYPROB	in P for all models if 1 side is certain and there is $O(1)$ number of uncertain agents		

Table 1. Summary of results.

We also consider two specific problems that are simpler than STABILITYPROBABILITY: (1) ISSTABILITYPROBABILITYNON-ZERO — For a given matching, is its stability probability non-zero? (2) ISSTABILITYPROBABILITYONE — For a given matching, is its stability probability one?

We additionally consider problems connected to, and more restricted than, MATCHINGWITHHIGHESTSTABILITYPROBABILITY: (1) EXISTSCERTAINLYSTABLEMATCHING — Does there exist a matching that has stability probability one? (2) EXISTSPOSSIBLYSTABLEMATCHING — Does there exist a matching that has non-zero stability probability?

Note that EXISTSPOSSIBLYSTABLEMATCHING is straightforward to answer for any of the three uncertainty models we consider here, since there exists a stable matching for each deterministic preference profile that is a possible realization of the uncertain preferences.

1.3 Results

Table 1 summarizes our main findings. Note that the complexity of each problem is considered with respect to the input size, and that under the lottery and joint probability models the size of the input could be exponential in n , namely $O(n! \cdot 2n)$ for the lottery model and $O((n!)^{2n})$ for the joint probability model, where n is the number of agents on either side of the market.

We point out that STABILITYPROBABILITY is #P-complete for the lottery model even when each agent has at most two possible preferences, but in P if one side has certain preferences. Additionally, we show that ISSTABILITYPROBABILITYNON-ZERO is in P for the lottery model if each agent has at most two possible preferences. Note that STABILITYPROBABILITY is open for the compact indifference model when both sides may be uncertain, and we also do not know the complexity of MATCHINGWITHHIGHESTSTABILITYPROBABILITY in the lottery model, except when only a constant number of agents are uncertain on the same side of the market.

2 Preliminaries

In the Stable Marriage problem, there are two sets of agents. Let M denote a set of n men and W a set of n women. We use the term *agents* when making statements that apply to both men and women, and the term *candidates* to refer to the agents on the opposite side of the market to that of an agent under consideration. Each agent has a linearly ordered preference over the candidates. An agent may be uncertain about his/her linear preference ordering. Let L denote the *uncertain preference profile* for all agents. We denote by $I = (M, W, L)$ an instance of a *Stable Marriage problem with Uncertain Linear Preferences (SMULP)*.

We say that a given uncertainty model is *independent* if any uncertain preference profile L under the model can be written as a product of uncertain preferences L_a for all agents a , where all L_a 's are independent. Note that the lottery and the compact indifference models are both independent, but the joint probability model is not.

A *matching* μ is a pairing of men and women such that each man is paired with at most one woman and vice versa; defining a list of (man, woman) pairs (m, w) . We use $\mu(m)$ to denote the woman w that is matched to m and $\mu(w)$ to denote the match for w . Given linearly ordered preferences, a matching is *stable* if there is no pair (m, w) not in μ where m prefers w to his current partner in μ , i.e., $w \succ_m \mu(m)$, and vice versa. If such a pair exists, it constitutes a *blocking pair*; as the pair would prefer to defect and match with each other rather than stay with their partner in μ . Given an instance of SMULP, a matching is *certainly stable* if it is stable with probability 1.

The following extensions of SM will come in handy in proving our results. The *Stable Marriage problem with Partially ordered lists (SMP)* is an extension of SM in which agents' preferences are partial orders over the candidates. The *Stable Marriage problem with Ties (SMT)* is a special case of SMP in which incomparability is transitive and is interpreted as indifference. Therefore, in SMT each agent partitions the candidates into different ties (equivalence classes), is indifferent between the candidates in the same tie, and has strict preference ordering over the ties. In some practical settings some agents may find some candidates unacceptable and prefer to remain unmatched than to get matched to the unacceptable ones. *SMP with Incomplete lists (SMPI)* and *SMT with Incomplete lists (SMTI)* captures these scenarios where each agent's partially ordered list contains only his/her acceptable candidates. A matching is *super-stable* in an instance of SMPI if it is stable w.r.t. all linear extensions of the partially ordered lists.

We define the *certainly preferred* relation \succ_a^{cert} for agent a . We write $b \succ_a^{\text{cert}} c$ if and only if agent a prefers b over c with probability 1. Based on the certainly preferred relation, we can define a dominance relation D : $D_m(w) = \{w\} \cup \{w' : w' \succ_m^{\text{cert}} w\}$; $D_w(m) = \{m\} \cup \{m' : m' \succ_w^{\text{cert}} m\}$. Based on the notion of the dominance relation, we present a useful characterization of certainly stable matchings for independent uncertainty models.

Lemma 1. *A matching μ is certainly stable for an independent uncertainty model if and only if for each pair $\{m, w\}$, $\mu(m) \in D_m(w)$ or $\mu(w) \in D_w(m)$.*

Proof. Assume that there exists a pair $\{m, w\}$ such that $\mu(m) \notin D_m(w)$ or $\mu(w) \notin D_w(m)$. Then, m has non-zero probability of preferring w over $\mu(m)$ and w has non-zero probability of preferring m over $\mu(w)$. But this means that μ has non-zero probability of not being stable.

Assume that a matching μ is certainly stable. Then no blocking pair $\{m, w\}$ has non-zero probability of forming. This is only possible if the pair $\{m, w\}$ is part of the matching or one of m and w have zero probability of preferring the blocking $\{m, w\}$ over their current match in μ . In either case, $\mu(m) \in D_m(w)$ or $\mu(w) \in D_w(m)$. \square

We point out that certainly preferred relation can be computed in polynomial time for all three models studied in this paper.

Certainly stable matchings are closely related to the notion of super-stable matchings [5, 9]. In fact we can define a certainly stable matching using a terminology similar to that of super-stability. Given a matching μ and an unmatched pair $\{m, w\}$, we say that $\{m, w\}$ *very weakly blocks (blocks)* μ if $\mu(m) \not\prec_m^{\text{cert}} w$ and $\mu(w) \not\prec_w^{\text{cert}} m$. The next claim then follows from Lemma 1.

Proposition 1. *A matching μ is certainly stable for an independent uncertainty model if and only if it admits no very weakly blocking pair.*

3 General Results

In this section, we first show that the complexity of all problems that we study are the same for complete and incomplete lists. We then present some general results that apply to multiple uncertainty models. We show that EXISTSCERTAINLYSTABLEMATCHING can be solved in polynomial time for any independent uncertainty model including lottery and compact indifference. We then prove that, when the number of uncertain agents is constant and one side of the market is certain, we can solve MATCHINGWITHHIGHESTSTABILITYPROBABILITY efficiently for each of the linear models.

3.1 The Case for Incomplete Lists

The next proposition explains that our efficient algorithms described for the case of complete lists can be extended to incomplete lists. Additionally, our hardness proofs for incomplete lists can be transformed for complete lists. In fact, all our hardness reductions, except Theorem 9, are for complete lists so they trivially extend to the case of incomplete lists.

Proposition 2. *The complexity of each computational problem studied in this paper are the same for complete and incomplete lists. Formally, if I is a linear model with incomplete lists then we can construct an instance I' with complete*

lists such that for each matching μ in I there exists a corresponding matching μ' in I' with $p(\mu, I) = p(\mu', I')$, such that μ can be obtained from μ' in polynomial time. Furthermore, μ is one of the most stable matchings in I if and only if the corresponding matching μ' is one of the most stable matching in I' . Therefore a polynomial time algorithm solving STABILITYPROBABILITY or MATCHINGWITHHIGHESTSTABILITYPROBABILITY for complete lists can be used to solve the same problem for incomplete lists in polynomial time.

Proof. In the case of complete lists we assumed that we have an equal number of men and women and everybody finds all candidates acceptable. When we consider the problem with incomplete lists we mean that the sizes of the two sets are not necessarily the same and not all the candidates are acceptable for the agents. However, we assume that in all realization of the preference profiles the same candidates are acceptable, so we only randomize on the preferences over the acceptable partners. Suppose that I is an instance of a probabilistic model with incomplete lists with sets M and W . Let us create the corresponding instance I' with sets M' and W' in the following way. First we ensure that $|M'| = |W'|$ by adding enough agents to the short side of the market. Then we complete the preference lists of each agent by adding the previously unacceptable candidates to the end of her/his list according to a predetermined order, e.g. by the indices of the agents. Suppose now that μ is a matching in I and X is the set of matched men in M , whilst $\mu(X) = Y$. Let us create a corresponding matching μ' in I' by extending μ with the unique stable matching for the subinstance restricted to the unmatched agents. Namely, let μ_u be the stable matching that matches $M' \setminus X$ to $W' \setminus Y$ in such a way that the k th pair contains the k th man and the k th woman from $M' \setminus X$ and $W' \setminus Y$, respectively according to their indices, and let $\mu' = \mu \cup \mu_u$. Now we claim that $p(\mu, I) = p(\mu', I')$. This is because there is no blocking pair in $(M' \setminus X) \times (W' \setminus Y)$, and any other pair is blocking for some preference profile in I if and only if it is blocking for the corresponding preference profile in I' , obviously. Furthermore, it is also clear that among the extensions of μ , μ' is the most stable one in I' . Therefore μ is one of the most stable matchings in I if and only if the corresponding extension, μ' is one of the most stable matchings in I' . Thus an efficient algorithm for STABILITYPROBABILITY or MATCHINGWITHHIGHESTSTABILITYPROBABILITY (or other subproblems) for complete lists can also be used to solve the same problems for the case of incomplete lists. This also implies that any hardness result proved for incomplete lists holds also for complete lists. \square

3.2 An Algorithm for the Lottery and Compact Indifference Models

Theorem 1. *For any independent uncertainty model in which the certainly preferred relation is transitive and can be computed in polynomial time, EXISTS CERTAINLY STABLE MATCHING can be solved in polynomial time.*

Proof. We prove this by reducing EXISTS CERTAINLY STABLE MATCHING to the problem of deciding whether an instance of SMP admits a super-stable matching

or not. The latter problem can be solved in polynomial time using algorithm SUPER-SMP in [13].

Let $I = (M, W, L)$ be an instance of EXISTSCERTAINLYSTABLEMATCHING under an independent uncertainty model, assuming that the certainly preferred relation is transitive and can be computed in polynomial time. We construct an instance $I' = (M, W, p)$ of SMP, in polynomial time, as follows. The set of men and women are unchanged. To create the partial preference ordering p_a for each agent a we do the following. W.l.o.g. assume that a is a man m . For every pair of women w_1 and w_2 (i) if $w_1 \succ_m^{\text{certain}} w_2$ then $(w_1, w_2) \in p_m$, denoting that m (strictly) prefers w_1 to w_2 in I' , (ii) if $w_2 \succ_m^{\text{certain}} w_1$ then $(w_2, w_1) \in p_m$, denoting that m (strictly) prefers w_2 to w_1 in I' . We claim, and show, that I' admits a super-stable matching if and only if I admits a matching with stability probability one. A matching μ is super-stable in I' if and only if it does not admit a very weak blocking pair. A pair (m, w) unmatched in μ is a very weak blocking pair if (i) m either prefers w to $\mu(m)$ or is indifferent between them, and (ii) w either prefers m to $\mu(w)$ or is indifferent between them. Agent a is indifferent between agents b and c under an SMP instance if neither (b, c) nor (c, b) is in p_a . It is easy to verify that an unmatched pair (m, w) in I' is a very weak blocking pair in μ if and only if $(\mu(m), w) \notin p_m$ and $(\mu(w), m) \notin p_w$.

Only if part: If I' admits a super-stable matching μ then μ is certainly stable in I . Assume for a contradiction that μ is not certainly stable in I . It then follows Lemma 1 that $\mu(m) \notin D_m(w)$ and $\mu(w) \notin D_w(m)$, implying that $\mu(m) \not\succ_m^{\text{strict}} w$ and $\mu(w) \not\succ_w^{\text{strict}} m$, and thus $(\mu(m), w) \notin p_m$ and $(\mu(w), m) \notin p_w$. Therefore (m, w) blocks μ in I' , a contradiction.

If part: If I admits a certainly stable matching μ then μ is super-stable in I' . Assume, for a contradiction, that μ is not super-stable in I' . Therefore there exists a very weak blocking pair (m, w) , implying that $(\mu(m), w) \notin p_m$ and $(\mu(w), m) \notin p_w$, which in turn implies that $\mu(m) \not\succ_m^{\text{strict}} w$ and $\mu(w) \not\succ_w^{\text{strict}} m$. The latter statement, coupled with the fact that m and w are not matched together, implies that $\mu(m) \notin D_m(w)$ and $\mu(w) \notin D_w(m)$, and thus by Lemma 1 μ is not stable in I , a contradiction. \square

3.3 An Algorithm for a Constant Number of Uncertain Agents

Theorem 2. *When the number of uncertain agents is constant and one side of the market is certain then MATCHINGWITHHIGHESTSTABILITYPROBABILITY is polynomial-time solvable for each of the linear models.*

Proof. Let $I = (M, W, L)$ be an instance of MATCHINGWITHHIGHESTSTABILITYPROBABILITY and let $X \subseteq M$ be the set of uncertain agents with $|X| = k$ for a constant k . We consider all the possible matchings between X and W , where their total number is $K = n(n-1) \dots (n-k)$. Let μ_i be such a matching for $i \in \{1 \dots K\}$. The main idea of the proof is to show that there exist an extension of μ_i to $M \cup W$ that has stability probability at least as high as any other extension of μ_i . In this way we will need to compute this probability for only a

polynomial number of matchings in n , that we can do efficiently for each model, and then compare them and select the one with the highest probability.

So we take a matching μ_i between sets X and W . Let $Y = \mu_i(X)$ (i.e., the partners of X in W) and let $M' = M \setminus X$ and $W' = W \setminus Y$. First, we compute the man-optimal matching μ_i^M for the sub-instance I' on $M' \cup W'$, that can be done efficiently by the Gale-Shapley algorithm [4]. Now, if there exist a blocking pair $\{m', w\}$ involving some certain agents $m \in M'$ and $w \in Y$ for $\mu \cup \mu_i^M$ in I , then we can conclude that any extension of μ_i for I will have zero probability of being stable. This is because any extension of μ_i for I that has a positive probability of being stable must also be stable for the sub-instance I' . If $\{m', w\}$ is a blocking pair for μ_i^M then it will remain blocking for any extension of μ_i for I as well, since w has the same partner and the m' cannot have a better partner either. Thus we can exclude the extensions of μ_i from the further consideration in this case.

Suppose now that there is no blocking pair of the form $\{m', w\}$, as explained above, for μ_i^M in I . We truncate the preference lists of men in M' in the following way. For each man $m' \in M'$ we remove all the women $w' \in W'$ from the list of m' that are less preferred by m' than some woman in Y that finds m' better than her partner in μ_i . That is, we remove w' from the list of m' if there exists $w \in Y$ such that $m' \succ_w \mu_i(w)$ and $w \succ_{m'} w'$. Let us denote the sub-instance for $M' \cup W'$ with the truncated lists as I_i^r . Now we compute the woman-optimal matching, μ_i^W in I_i^r . Let $\mu_i^* = \mu_i \cup \mu_i^W$ be the extended matching in I . This is stable for the certain agents by the construction.

Finally, we will show that for any matching μ'_i , that is an extension of μ_i to I , the stability probability of μ'_i is less than, or equal to, the stability probability of μ_i^* . If μ'_i is not stable for the certain agents then μ'_i has zero probability of being stable, thus the statement holds. If μ'_i is stable for the certain agents then it must also be stable in I_i^r , and each woman in W' weakly prefers her partner in μ'_i to her partner in μ'_i , since she gets her optimal stable partner for I_i^r in μ_i^* . Therefore, if μ'_i is stable under a preference profile then μ_i^* will also be stable, so the statement follows. Thus, there remain only a polynomial number (K) of candidate matchings in n for which we have to compute the probabilities. STABILITYPROBABILITY is polynomial-time solvable for all the three models we consider given that one side has certain preferences, as described in Theorems 3, 8, and 10. \square

4 Lottery Model

In this section we focus on the lottery model.

Theorem 3. *For the lottery model, if one side has certain preferences, STABILITYPROBABILITY is polynomial-time solvable.*

Proof. Without loss of generality, assume that men have certain preferences. The following procedure gives us the stability probability of μ for any given μ .
(1) For each uncertain woman w identify those preferences that allow her not

to form a blocking pair. We can do this in polynomial time as men have strict preferences and therefore for each preference ordering of w we only need to look up the (one and only) preference ordering of each m who w prefers to $\mu(w)$. (2) For each uncertain woman w , add up the probabilities of all preference orderings that pass the test in the first step. (3) multiply the added-up probabilities for all w obtained in step (2). \square

Theorem 4. *For the lottery model, ISSTABILITYPROBABILITYONE can be solved in linear time.*

Proof. The problem is equivalent to checking whether the given matching μ has non-zero probability of *not* being stable. This can be checked as follows. For each possible pair of agents $\{m, w\}$ that are not matched to each other, we check whether they can form a blocking pair with non-zero probability. For this, we just need to check whether m prefers w in some possible preference over $\mu(m)$ and whether w prefers m in some possible preference over $\mu(w)$. \square

Theorem 5. *For the lottery model, ISSTABILITYPROBABILITYNON-ZERO is polynomial-time solvable when each agent has at most two possible preference orderings.*

Proof. The problem is to decide whether there is some preference ordering for each agent (among the ones in their lottery) such that the given matching is stable. If each agent has at most two possible preference orderings in their lottery, we can reduce the problem to an instance φ of 2SAT, as follows.

Let $\{a_1, \dots, a_n\}$ and $\{b_1, \dots, b_n\}$ be the two sets of agents. Moreover, for each agent c and each $i \in \{1, 2\}$, let $\text{pref}(c, i)$ denote the i -th preference in the lottery for agent c .

We introduce a propositional variable for each preference $\text{pref}(c, i)$ —which we also call $\text{pref}(c, i)$. Intuitively, these variables indicate which preference for the agents we choose to make the matching stable.

For each agent c , we add the following clauses to φ , to ensure that for each agent c there is exactly one preference that is selected: $(\text{pref}(c, 1) \vee \text{pref}(c, 2)) \wedge (\neg \text{pref}(c, 1) \vee \neg \text{pref}(c, 2))$.

Then, we add clauses to ensure that the selected matching is stable. For each agent c and each $i \in \{1, 2\}$, let $B_{c,i}$ be the set of preferences $\text{pref}(c', i')$ —for $c' \neq c$ and $i' \in \{1, 2\}$ —such that $\text{pref}(c, i)$ and $\text{pref}(c', i')$ together lead to the given matching being unstable (with (c, c') being a blocking pair). Then, for each c, i , we add the following clauses: $(\neg \text{pref}(c, i) \vee \neg \text{pref}(c', i'))$ for each $\text{pref}(c', i') \in B_{c,i}$.

The given matching is then stable if and only if φ is satisfiable. Since φ is a 2CNF, this can be decided in linear time. \square

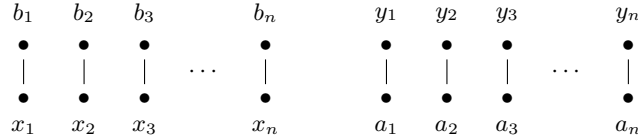
Theorem 6. *For the lottery model, STABILITYPROBABILITY is $\#P$ -complete, even when each agent has at most two possible preferences.*

Proof. We show how to count the number of satisfying assignments for a 2CNF formula using the problem STABILITYPROBABILITY for the lottery model where each agent has two possible preferences. Since this problem is #P-hard, we get #P-hardness also for STABILITYPROBABILITY.

Let φ be a 2CNF formula over the variables x_1, \dots, x_n . We firstly transform φ to a 2CNF formula φ' over the variables $x_1, \dots, x_n, y_1, \dots, y_n$ that has exactly the same number of satisfying assignments, and that satisfies the property that each clause contains one variable among x_1, \dots, x_n and one variable among y_1, \dots, y_n . We do so as follows. Firstly, for each $1 \leq i \leq n$, we add the clauses $(\neg x_i \vee y_i)$ and $(\neg y_i \vee x_i)$, ensuring that in each satisfying assignment the variables x_i and y_i get assigned the same truth value. Then, for each clause of φ , we replace one occurrence of a variable among x_1, \dots, x_n by the corresponding variable among y_1, \dots, y_n , and we add the resulting clause to φ' . For example, if φ contains the clause $(x_1 \vee \neg x_3)$, we would add the clause $(x_1 \vee \neg y_3)$ to φ' . It is readily verified that φ' has the same number of satisfying assignments as φ .

Moreover, we may assume without loss of generality that for any two variables of φ' , there is at most one clause of φ' that contains both of these variables. If in φ there are two variables x_1 and x_2 and clauses $(x_1 \vee x_2)$ and $(\neg x_1 \vee \neg x_2)$, for instance, we can construct φ' to contain the clauses $(x_1 \vee y_2)$ and $(\neg y_1 \vee \neg x_2)$.

We now construct an instance of STABILITYPROBABILITY. The sets of agents that we consider are $\{x_1, \dots, x_n, a_1, \dots, a_n\}$ and $\{y_1, \dots, y_n, b_1, \dots, b_n\}$. The matching that we consider matches x_i to b_i and matches y_i to a_i , for each $1 \leq i \leq n$. This is depicted below. Each agent b_i has only a single possible preference, namely one where they prefer x_i over all other agents. Similarly, each agent a_i has a single possible preference where they prefer y_i over all other agents. In other words, the agents a_i and b_i are perfectly happy with the given matching. The agents x_i and y_i each have two possible preferences, that are each chosen



with probability $\frac{1}{2}$. These two possible preferences are associated with setting these variables to true or false, respectively. We describe how these preferences are constructed for the agents x_i . The construction for the preferences of the agents y_i is then entirely analogous.

Take an arbitrary agent x_i . We show how to construct the two possible preferences for agent x_i , which we denote by p_{x_i} and $p_{\neg x_i}$. Both of these possible preferences are based on the following partial ranking: $b_1 > b_2 > \dots > b_n$, and we add some of the agents y_1, \dots, y_n to the top of this partial ranking, and the remaining agents to the bottom of this partial ranking.

To the ranking p_{x_i} we add exactly those agents y_j to the top where φ' contains a clause $(\neg x_i \vee y_j)$ or a clause $(\neg x_i \vee \neg y_j)$. All remaining agents we add to the bottom. Similarly, to the ranking $p_{\neg x_i}$ we add exactly those agents y_j to the top where φ' contains a clause $(x_i \vee y_j)$ or a clause $(x_i \vee \neg y_j)$. The rankings p_{y_i} and $p_{\neg y_i}$, for the agents y_i , are constructed entirely similarly.

Now consider a truth assignment $\alpha : \{x_1, \dots, x_n, y_1, \dots, y_n\} \rightarrow \{0, 1\}$, and consider the corresponding choice of preferences for the agents $x_1, \dots, x_n, y_1, \dots, y_n$, where for each agent x_i the preference p_{x_i} is chosen if and only if $\alpha(x_i) = 1$, and for each agent y_i the preference p_{y_i} is chosen if and only if $\alpha(y_i) = 1$. Then α satisfies φ' if and only if the corresponding choice of preferences leads to the matching being stable. Since each combination of preferences is equally likely to occur, and there are 2^{2n} many combinations of preferences, the probability that the given matching is stable is exactly $q = \frac{s}{2^{2n}}$, where s is the number of satisfying truth assignments for φ . Therefore, given q , s can be obtained by computing $s = q2^{2n}$. \square

If each agent is allowed to have three possible preferences, then even the following problem is NP-complete. The statement can be proved via a reduction from Exact Cover by 3-Sets (X3C).

Theorem 7. *For the lottery model, ISSTABILITYPROBABILITYNON-ZERO is NP-complete.*

Proof. The problem is in NP, since we only need to provide one profile that occurs with non-zero probability for which the given matching is stable. We show NP-hardness by giving a reduction from Exact Cover by 3-Sets (X3C). Let (X, C) be an instance of X3C, where $|X| = 3n$ for some n , and $C = \{c_1, \dots, c_m\}$ is a collection of sets $c_i \subseteq X$, each of size 3. Moreover, let $c_i = \{x_{\ell_{i,1}}, x_{\ell_{i,2}}, x_{\ell_{i,3}}\}$, for each $1 \leq i \leq m$. The problem is to decide whether there is a subset $C' \subseteq C$ of size exactly n such that $\bigcup C' = X$.

We construct an instance of our problem as follows. We let $\{a_1, \dots, a_n, a'_1, \dots, a'_{3n}\}$ and $\{b_1, \dots, b_n, b'_1, \dots, b'_{3n}\}$ be the two sets of agents, we match a_i to b_i —for each $1 \leq i \leq n$ —and we match a'_j to b'_j —for each $1 \leq j \leq 3n$. This is depicted below.

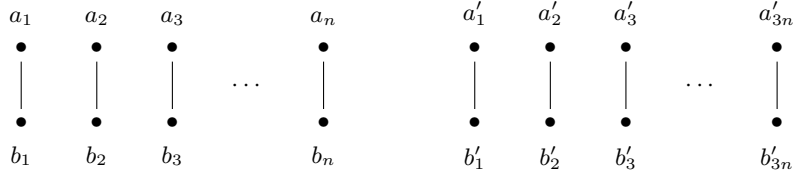


Fig. 1. Illustration of the reduction

Each agent a_i prefers their matching to b_i over any other possible match, i.e., agent a_i has one preference, where b_i is ranked first, and the rest of the agents appear in some fixed order after b_i .

Similarly, each agent b'_j prefers their matching to a'_j over any other possible match. That is, agent b'_j has one preference ordering in which a'_j is ranked first and the rest of the agents appear in some fixed order after a'_j .

Then, for each agent b_i , we add the following $|C|$ possible preferences to the lottery:

$$\begin{aligned} P_{i,1} : & \quad a'_{\ell_{1,1}} > a'_{\ell_{1,2}} > a'_{\ell_{1,3}} > a_i > \dots \\ P_{i,2} : & \quad a'_{\ell_{2,1}} > a'_{\ell_{2,2}} > a'_{\ell_{2,3}} > a_i > \dots \\ & \quad \vdots \\ P_{i,m} : & \quad a'_{\ell_{m,1}} > a'_{\ell_{m,2}} > a'_{\ell_{m,3}} > a_i > \dots \end{aligned}$$

where in each preference the remaining agents appear in any (fixed) order after a_i . In other words, b_i prefers three agents a'_j to their current match, and these three form some set $c \in C$.

Finally, for each agent a'_j , we add the following n possible preferences to the lottery:

$$\begin{aligned} P'_{j,1} : & \quad b_2 > \dots > b_n > b'_j > b_1 > b'_1 > \dots > b'_{j-1} > b'_{j+1} > \dots > b'_{3n} \\ P'_{j,2} : & \quad b_1 > b_3 > \dots > b_n > b'_j > b_2 > b'_1 > \dots > b'_{j-1} > b'_{j+1} > \dots > b'_{3n} \\ P'_{j,3} : & \quad b_1 > b_2 > b_4 > \dots > b_n > b'_j > b_3 > b'_1 > \dots > b'_{j-1} > b'_{j+1} > \dots > b'_{3n} \\ & \quad \vdots \\ P'_{j,n} : & \quad b_1 > \dots > b_{n-1} > b'_j > b_n > b'_1 > \dots > b'_{j-1} > b'_{j+1} > \dots > b'_{3n} \end{aligned}$$

That is, each agent a'_j prefers each of the agents b_1, \dots, b_n , except one, to their current match (and they never prefer any of the agents $b'_{j'}$ for $j' \neq j$ over their current match).

We can show that there is a choice of preferences for the agents that makes this matching stable if and only if $(X, C) \in \text{X3C}$.

(\Rightarrow) Firstly, suppose that there is a choice of preferences for the agents that makes this matching stable. That is, for each agent b_i there is some preference ordering P_{i,ℓ_i} , and for each agent a'_j there is some preference ordering P'_{j,k_j} , such that these orderings (together with the fixed preference orderings for the agents a_i and b'_j) make this matching stable. Now, consider the set $C' = \{c_\ell : i \in [n], \ell = \ell_i\}$. We show that $\bigcup C' = X$. To derive a contradiction, suppose that this is not the case, that is, suppose that $\bigcup C' \neq X$. Then, since $|C'| = n$, $|X| = 3n$ and for each $c \in C'$ it holds that $|c| = 3$, we know that there must be some $c_\ell, c_{\ell'} \in C'$ such that $c_\ell \cap c_{\ell'} \neq \emptyset$. Say that $x_j \in c_\ell \cap c_{\ell'}$. Therefore, there must be some $i, i' \in [n]$ such that both b_i and $b_{i'}$ prefer a'_j over their current match. On the other hand, a'_j will prefer either b_i or $b_{i'}$ over their current match. Therefore, either b_i and a'_j or $b_{i'}$ and a'_j will form a blocking pair. Thus, the matching is not stable. From this we can conclude that $\bigcup C' = X$.

(\Leftarrow) Conversely, suppose that there exists some $C' \subseteq C$ of size exactly n such that $\bigcup C' = X$. Let $C' = \{c_{\ell_1}, \dots, c_{\ell_n}\}$. Now, for each agent b_i we pick some preference ordering, and for each agent a'_j we pick some preference ordering, such that these orderings (together with the fixed preference orderings for the agents a_i and b'_j) make the matching stable. For each agent b_i , we pick the preference ordering P_{i,ℓ_i} , and for each agent a'_j we pick the preference ordering P'_{j,k_j} , where $k_j \in [n]$ is the unique value such that $x_j \in c_{\ell_{k_j}}$. It is straightforward to verify that these preferences make the matching stable. \square

We obtain the first corollary from Theorem 7 and the second from [16, Proposition 8] and Theorem 7.

Corollary 1. *For the lottery model, unless $P = NP$, there exists no polynomial-time algorithm for approximating STABILITYPROBABILITY of a given matching.*

Corollary 2. *For the lottery model, unless $NP = RP$, there is no FPRAS for STABILITYPROBABILITY.*

5 Compact Indifference Model

The compact indifference model is equivalent to assuming that we are given an instance of SMT and each linear order over candidates (each possible preference ordering) is achieved by breaking ties independently at random with uniform probabilities. It is easy to show that ISSTABILITYPROBABILITYNONZERO, ISSTABILITYPROBABILITYONE, and EXISTS CERTAINLY STABLE MATCHING are all in P.

Proposition 3. *For the compact indifference model, ISSTABILITYPROBABILITYNONZERO is in P.*

Proof. This is equivalent to checking whether a given matching μ is weakly stable in the given SMTI instance. To check this we only have to look for a blocking pair, which can be done in polynomial time: take every possible pair (m, w) who are not matched together and check whether they both strictly prefer each other to their current partner. \square

Proposition 4. *For the compact indifference model, ISSTABILITYPROBABILITYONE is in P.*

Proof. The problem is polynomial-time solvable. We go through all the blocking pairs and check if any blocking pair is feasible. For each blocking pair, we break ties (if there are any) in favour of the blocking pair. Given that we break ties in favour of the blocking pairs, if there exists a blocking pair that is feasible, the stability probability is not one. \square

Proposition 5. *For the compact indifference model, EXISTS CERTAINLY STABLE MATCHING is in P.*

Proof. Deciding whether there is matching that is stable with probability one is equivalent to deciding whether there is a matching that is stable w.r.t. all refinements, a super-stable matching. Given an instance of SMTI one can decide in polynomial time whether it admits a super-stable matching or not [12]. \square

We do not yet know the complexity of computing the stability probability of a given matching under the compact indifference model, but this problem can be shown to be in P if one side has certain preferences.

Theorem 8. *In the compact indifference model, if one side has certain preferences, STABILITYPROBABILITY is polynomial-time solvable.*

Proof. Assume, w.l.o.g., that men have certain preferences. The following procedure gives us the stability probability of any given matching μ . (1) For each uncertain woman w identify those men with whom she can potentially form a blocking pair. That is, those m such that $w \succ_m \mu(m)$ and w is indifferent between m and her partner in μ . Assume there are k of such men. The probability of w not forming a blocking pair with any men is then $\frac{1}{k+1}$. (2) Multiply the probabilities from step 1. \square

We next show that MATCHINGWITHHIGHESTSTABILITYPROBABILITY is NP-hard. For an instance I of SMT and matching μ , let $p(\mu, I)$ denote the probability of μ being stable, and let $p_S(I) = \max\{p(\mu, I) \mid \mu \text{ is a matching in } I\}$, that is the maximum probability of a matching being stable. A matching μ is said to be weakly stable if there exists a tie-breaking rule where μ is stable. Therefore a matching μ has positive probability of being stable if and only if it is weakly stable. Furthermore, if the number of possible tie-breaking is N then any weakly stable matching has a probability of being stable at least $\frac{1}{N}$.

An extreme case occurs if we have one woman only with n men, where the woman is indifferent between all men. In this case any matching (pair) has a $\frac{1}{n}$ probability of being stable. An even more unfortunate scenario is when we have n men and n women, each women is indifferent between all men, and each man ranks the women in a strict order in the same way, e.g. in the order of their indices. In this case, the probability that the first woman picks her best partner, and thus does not block any matching is $\frac{1}{n}$. Suppose that the first woman picked her best partner, the probability that the second woman also picks her best partner from the remaining $n - 1$ men is $\frac{1}{n-1}$, and so on. Therefore, the probability that an arbitrary complete matching is stable is $\frac{1}{n(n-1)\dots 2} = \frac{1}{n!}$.

Theorem 9. *For the compact indifference model MATCHINGWITHHIGHEST-STABILITYPROBABILITY is NP-hard, even if only one side of the market has uncertain agents.*

Proof. For an instance I of SMTI, let $opt(I)$ denote the maximum size of a weakly stable matching in I . Halldorsson et al. [7] showed [in the proof of Corollary 3.4] that given an instance I of SMTI of size n , where only one side of the market has agents with indifferences and each of these agents has a single tie of size

two, and any arbitrary small positive ϵ , it is NP-hard to distinguish between the following two cases: (1) $\text{opt}(I) \geq \frac{21-\epsilon}{27}n$ and (2) $\text{opt}(I) < \frac{19+\epsilon}{27}n$.

When choosing ϵ so that $0 < \epsilon < \frac{1}{2}$ we can simplify the above cases to (1) $\text{opt}(I) > \frac{41}{54}n$, since $\text{opt}(I) \geq \frac{21-\epsilon}{27}n > \frac{41}{54}n$ and (2) $\text{opt}(I) < \frac{39}{54}n$, since $\text{opt}(I) < \frac{19+\epsilon}{27}n < \frac{39}{54}n$.

Therefore, the number of agents left unmatched on either side of the market is less than $\frac{13}{54}n$ in the first case and more than $\frac{15}{54}n$ in the second case. Let us now extend instance I to a larger instance of SMTI I' as follows. Besides the n men $M = \{m_1, \dots, m_n\}$ and n women $W = \{w_1, \dots, w_n\}$, we introduce $\frac{13}{54}n$ men $X = \{x_1, \dots, x_k\}$ and another $\frac{n}{27}$ men $Y = \{y_1, \dots, y_l\}$ and $\frac{n}{27}$ women $Z = \{z_1, \dots, z_l\}$. Furthermore, for each $y_j \in Y$, we introduce n men $Y^j = \{y_1^j, \dots, y_n^j\}$. We create the preferences of I' as follows. The preferences of men M remain the same. For each woman $w \in W$ we append the men X and then Y at the end of her list in the order of their indices. Each man $x_i \in X$ has only all the women W in his list in the order of their indices. Furthermore, each $y_j \in Y$ has all the women W first in his preference list in the order of their indices and then z_j . Let each $z_j \in Z$ has y_j as first choice and then all the men Y^j in one tie of size n . Each man in Y^j has only z_j in his list. We will show that in case one $p_S(I') \geq \frac{1}{2^n}$, whilst in case two $p_S \leq (\frac{1}{n})^{\frac{n}{27}}$. Therefore, for $n > 2^{27}$, it is NP-hard to decide which of the two separate intervals contains the value $p_S(I')$.

To show the above statement, suppose first that we have the first case, so $\text{opt}(I) > \frac{41}{54}n$ and therefore less than $\frac{13}{54}n$ women are left unmatched in a maximum size weakly stable matching μ for I , denoted by $W_u \subset W$. We extend μ to μ' for I' as follows. We assign all the women in W_u to men in X in the unique stable way, namely we pair them in a mutually increasing order of their indices. Since $|X| > |W_u|$, we now matched all women in W , and left some men in X unmatched in μ' . We complete the matching by assigning y_j to z_j for each $j = 1, \dots, n$ and leaving all of the men in Y^j for all j unmatched. We shall see that no matter how we break the ties in I' , blocking pair can appear between the original I agents only, and therefore the probability of μ' being stable in I' is the same as the probability of μ being stable in I . Since we have at most n ties in I , each of length two, the number of different tie-breakings is at most 2^n , out of which at least one is stable. Therefore $p(\mu, I') = p(\mu, I) \geq \frac{1}{2^n}$.

In the second case, $\text{opt}(I) < \frac{39}{54}n$ and therefore more than $\frac{15}{54}n$ women are left unmatched in any weakly stable matching μ for I . Let μ' be one of the most stable matchings in I' . First we have to note that the restriction of μ' to I must be weakly stable in I , since otherwise $p(\mu', I') = 0$. Let W_u denote the set of women that are not matched to any man from M in μ' . According to our assumption $|W_u| > \frac{15}{54}n$, whilst $|X| + |Y| = \frac{15}{54}n$, therefore in order to avoid a certain blocking pair between W_u and $X \cup Y$ we shall match all the men in $X \cup Y$ to women in W_u in the only stable way (in the order of indices, where men in X are coming before men in Y), and leaving some women in W_u unmatched in μ' . However, in this case no agent $z_j \in Z$ can be matched to y_j , and therefore, even if there was no potential blocking pair between agents of I , the probability that z_j is matched the best partner from Y^j is $\frac{1}{n}$ independently for each $z_j \in Z$.

Therefore the probability of μ' being stable is at most $(\frac{1}{n})^{\frac{n}{27}}$, which completes the proof of the first statement.

Regarding the NP-hardness of finding one of the most stable matchings, we shall prove that we can decide between the two cases according to the number of unmatched women in W in the restriction of μ' to I , where μ' is one of the most stable matchings in I' . To see this, let W_u denote again the set of women that are not matched to any man in M under μ' . In the first case, when $\text{opt}(I) > \frac{41}{54}n$, it must be the case that $|W_u| < \frac{15}{54}n$, since otherwise $p(\mu', I)$ would be less than $(\frac{1}{n})^{\frac{n}{27}}$ and could not achieve $\frac{1}{2^n}$, that is the minimum value for $p_S(I')$, as shown in the above argument. Whilst, in the second case $|W_u| > \frac{15}{54}n$ must hold, since $\text{opt}(I) < \frac{39}{54}n$ was assumed. \square

6 Joint Probability Model

In this section, we examine problems concerning the joint probability model.

Theorem 10. *For the joint probability model, STABILITYPROBABILITY can be solved in polynomial time.*

Proof. The probability that a given matching is stable is equivalent to the probability weight of the preference profiles for which the matching is stable. This can be checked as follows. We check the preference profiles for which the given matching is stable (for one profile, this can be checked in $O(n^2)$). Then we add the probabilities of those profiles for which the matching is stable. The sum of the probabilities is the probability that the matching is stable. \square

Corollary 3. *For the joint probability model, ISSTABILITYPROBABILITYNONZERO and ISSTABILITYPROBABILITYONE can be solved in polynomial time.*

For the joint probability model, the problem EXISTSCERTAINLYSTABLEMATCHING is equivalent to checking whether the intersection of the sets of stable matchings of the different preference profiles is empty or not.

Theorem 11. *For the joint probability model, EXISTSCERTAINLYSTABLEMATCHING is NP-complete.*

Proof. The problem is in NP, since computing STABILITYPROBABILITY can be done in polynomial time by Theorem 10. The NP-hardness proof is by reduction from 3-Colorability. Let $G = (V, E)$ be a graph specifying an instance of 3-Colorability, where $V = \{v_1, \dots, v_n\}$. We construct an instance I of SMULP assuming the joint probability model.

For each vertex $v_i \in V$, we introduce three men $m_{i,1}, m_{i,2}, m_{i,3}$ and three women $w_{i,1}, w_{i,2}, w_{i,3}$. Then, we introduce one preference profile P_0 that ensures that every certainly stable matching matches—for each $i \in [n]$ —each $m_{i,j}$ to some $w_{i,j'}$ and, vice versa, each $w_{i,j}$ to some $m_{i,j'}$, for $j, j' \in [3]$. Moreover,

it ensures that for each $i \in [n]$, exactly one of three matchings between the men $m_{i,j}$ and the women $w_{i,j}$ must be used:

- (1) $m_{i,1}$ is matched to $w_{i,1}$, $m_{i,2}$ is matched to $w_{i,2}$, and $m_{i,3}$ is matched to $w_{i,3}$;
- (2) $m_{i,1}$ is matched to $w_{i,2}$, $m_{i,2}$ is matched to $w_{i,3}$, and $m_{i,3}$ is matched to $w_{i,1}$; or
- (3) $m_{i,1}$ is matched to $w_{i,3}$, $m_{i,2}$ is matched to $w_{i,1}$, and $m_{i,3}$ is matched to $w_{i,2}$;

Intuitively, choosing one of the matchings (1)–(3) for the agents $m_{i,j}, w_{i,j}$ corresponds to coloring vertex v_i with one of the three colors in $\{1, 2, 3\}$.

Then, for each edge $e = \{v_{i_1}, v_{i_2}\} \in E$, and for each color $c \in \{1, 2, 3\}$, we introduce a preference profile $P_{e,c}$ that ensures that in any certainly stable matching, the agents $m_{i_1,j}, w_{i_1,j}$ and the agents $m_{i_2,j}, w_{i_2,j}$ cannot both be matched to each other with matching (c) . We let each preference profile appear with non-zero probability (e.g., we take a uniform lottery over all the preference profiles that we introduced). As a result, any certainly stable matching directly corresponds to a proper 3-coloring of G .

A detailed description of the preference profiles P_0 and $P_{e,c}$ and a proof of correctness for this reduction follows.

In P_0 , for each $i \in [n]$, the preferences for $m_{i,j}, w_{i,j}$ are as follows:

$$\begin{array}{ll}
m_{i,1} : & w_{i,1}, w_{i,2}, w_{i,3}, - - - & w_{i,1} : & m_{i,2}, m_{i,3}, m_{i,1}, - - - \\
m_{i,2} : & w_{i,2}, w_{i,3}, w_{i,1}, - - - & w_{i,2} : & m_{i,3}, m_{i,1}, m_{i,2}, - - - \\
m_{i,3} : & w_{i,3}, w_{i,1}, w_{i,2}, - - - & w_{i,3} : & m_{i,1}, m_{i,2}, m_{i,3}, - - -
\end{array}$$

Next, we continue with the preference profiles $P_{e,c}$. Take an arbitrary $e = \{v_{i_1}, v_{i_2}\} \in E$ and an arbitrary $c \in \{1, 2, 3\}$. In $P_{e,c}$, the preferences for $m_{i,j}, w_{i,j}$ for each $i \in [n] \setminus \{i_1, i_2\}$ are exactly the same as in P_0 . Only the preferences for $m_{i_1,j}, w_{i_1,j}$ and $m_{i_2,j}, w_{i_2,j}$ differ from P_0 ; namely, we construct these preferences as follows.

For $m_{i_1,j}, w_{i_1,j}$, we start with preferences that (i) for all $m_{i_1,j}$ have $w_{i_1,1}, w_{i_1,2}, w_{i_1,3}$ as top three choices, (ii) for all $w_{i_1,j}$ have $m_{i_1,1}, m_{i_1,2}, m_{i_1,3}$ as top three choices, (iii) admit only matchings (1), (2), and (3) as stable matchings between the agents $m_{i_1,j}, w_{i_1,j}$, and (iv) for the men $m_{i_1,j}$ the matching (c) is the worst option among the matchings (1), (2), and (3). Similarly, for $m_{i_2,j}, w_{i_2,j}$, we start with preferences that satisfy conditions (i), (ii) and (iii), and additionally satisfy the condition (iv') that for the women $w_{i_2,j}$ the matching (c) is the worst option among the matchings (1), (2), and (3). Then, we modify the preferences for $m_{i_1,1}$ and $w_{i_2,1}$ slightly. For $m_{i_1,1}$, we insert $w_{i_2,1}$ between his second and third preferred woman. Similarly, for $w_{i_2,1}$, we insert $m_{i_1,1}$ between her second and third preferred man. As a result, $m_{i_1,1}$ and $w_{i_2,1}$ form a blocking pair in this preference profile if both the agents $m_{i_1,j}, w_{i_1,j}$ and the agents $m_{i_2,j}, w_{i_2,j}$ are matched to each other using matching (c) —and not if either set of agents is matched to each other using some other matching (c') .

For example, consider $e = \{v_{i_1}, v_{i_2}\}$ and $c = 2$. The preferences for the agents $m_{i_1,j}, w_{i_1,j}$ and $m_{i_2,j}, w_{i_2,j}$ in the preference profile $P_{e,c}$ are as follows:

$m_{i_1,1} :$	$w_{i_1,1}, w_{i_1,3}, \mathbf{w}_{i_2,1}, w_{i_1,2}, - - -$	$m_{i_2,1} :$	$w_{i_2,2}, w_{i_2,3}, w_{i_2,1}, - - -$
$m_{i_1,2} :$	$w_{i_1,2}, w_{i_1,1}, w_{i_1,3}, - - -$	$m_{i_2,2} :$	$w_{i_2,3}, w_{i_2,1}, w_{i_2,2}, - - -$
$m_{i_1,3} :$	$w_{i_1,3}, w_{i_1,2}, w_{i_1,1}, - - -$	$m_{i_2,3} :$	$w_{i_2,1}, w_{i_2,2}, w_{i_2,3}, - - -$
$w_{i_1,1} :$	$m_{i_1,3}, m_{i_1,2}, m_{i_1,1}, - - -$	$w_{i_2,1} :$	$m_{i_2,1}, m_{i_2,2}, \mathbf{m}_{i_1,1}, m_{i_2,3}, - - -$
$w_{i_1,2} :$	$m_{i_1,1}, m_{i_1,3}, m_{i_1,2}, - - -$	$w_{i_2,2} :$	$m_{i_2,2}, m_{i_2,3}, m_{i_2,1}, - - -$
$w_{i_1,3} :$	$m_{i_1,2}, m_{i_1,1}, m_{i_1,3}, - - -$	$w_{i_2,3} :$	$m_{i_2,3}, m_{i_2,1}, m_{i_2,2}, - - -$

We argue that G has a proper 3-coloring if and only if there is a certainly stable matching for the probability distribution over preference profiles that we constructed.

(\Rightarrow) Firstly, suppose that G has a proper 3-coloring, say $\chi : V \rightarrow \{1, 2, 3\}$. We can then construct a certainly stable matching as follows. For each $i \in [n]$, we match the agents $m_{i,j}, w_{i,j}$ to each other using matching (c_i) , where $c_i = \chi(v_i)$. Clearly, this matching is stable for P_0 . Moreover, because χ is a proper 3-coloring of G , it is straightforward to verify that this matching is also stable for each $P_{e,c}$.

(\Leftarrow) Conversely, suppose that there is a certainly stable matching. We know that in this matching, each man $m_{i,j}$ must be matched to some woman $w_{i,j'}$, and vice versa, each woman $w_{i,j}$ must be matched to some man $m_{i,j'}$. If this were not the case, the matching would not be stable for P_0 , and thus not certainly stable. Moreover, by a similar argument, we know that for each $i \in [n]$, the matching between the men $m_{i,j}$ and the women $w_{i,j}$ must be one of the matchings (1), (2), or (3). We can then construct a 3-coloring $\chi : V \rightarrow \{1, 2, 3\}$ as follows. For each $i \in [n]$, we let $\chi(v_i) = c_i$, where (c_i) is the matching used in the certainly stable matching to match the men $m_{i,j}$ and the women $w_{i,j}$ to each other.

We argue that χ is a proper 3-coloring of G . Suppose that this is not the case, that is, that there is some $e = \{v_{i_1}, v_{i_2}\}$ such that $\chi(v_{i_1}) = \chi(v_{i_2}) = c$. Now consider the preference profile $P_{e,c}$. By construction of χ , we know that in the certainly stable matching, both the agents $m_{i_1,j}, w_{i_1,j}$ and the agents $m_{i_2,j}, w_{i_2,j}$ are matched to each other using matching (c) . However, then by construction of $P_{e,c}$, $m_{i_1,1}$ and $w_{i_2,1}$ form a blocking pair in $P_{e,c}$. This is a contradiction with our assumption that the matching we considered is certainly stable. From this, we can conclude that χ is a proper 3-coloring of G . \square

By modifying the proof of Theorem 11, the following can also be proved.

Corollary 4. *For the joint probability model, EXISTS CERTAINLY STABLE-MATCHING is NP-complete, even when there are only 16 preference profiles in the lottery.*

Proof. We show this by modifying the proof of Theorem 11. We know that 3-Colorability is NP-hard already when restricted to graphs of degree 4 [1]. We use the reduction in the proof of Theorem 11, and we assume that the given graph G has degree 4. Then, by Vizing's Theorem [15], we know that we can give a proper edge coloring of G that uses at most 5 colors. Moreover, we can find

such an edge coloring in polynomial time. Then, since in the proof of Theorem 11, in each preference profile $P_{e,c}$ with $e = \{v_{i_1}, v_{i_2}\}$, only the preferences for the agents $m_{i_1,j}, w_{i_1,j}, m_{i_2,j}, w_{i_2,j}$ differ from P_0 , we can, for each color $c \in \{1, 2, 3\}$, combine the preference profiles $P_{e,c}$ for all edges e that are colored with the same color. This results in only 16 preference profiles: P_0 , and a preference profile for each of the 5 edge colors and each of the 3 vertex colors. \square

7 Future work

First we note that we left open two outstanding questions, as described in Table 1. In this paper we focused on the problem of computing a matching with the highest stability probability. However, a similarly reasonable goal could be to minimize the expected number of blocking pairs. It would also be interesting to investigate some further realistic probability models, such as the situation when the candidates are ranked according to some noisy scores (like the SAT scores in the US college admissions). This would be a special case of the joint probability model that may turn out to be easier to solve. Finally, in a follow-up paper we are planning to investigate another probabilistic model that is based on independent pairwise comparisons.

Acknowledgments. Biró is supported by the Hungarian Academy of Sciences under its Momentum Programme (LP2016-3) and the Hungarian Scientific Research Fund, OTKA, Grant No. K108673. Rastegari was supported EPSRC grant EP/K010042/1 at the time of the submission. The authors gratefully acknowledge the support from European Cooperation in Science and Technology (COST) action IC1205. Serge Gaspers is the recipient of an Australian Research Council (ARC) Future Fellowship (FT140100048) and acknowledges support under the ARC’s Discovery Projects funding scheme (DP150101134). NICTA is funded by the Australian Government through the Department of Communications and the ARC through the ICT Centre of Excellence Program.

References

1. D. P. Dailey. Uniqueness of colorability and colorability of planar 4-regular graphs are NP-complete. *Discrete Mathematics*, 30(3):289–293, 1980.
2. J. Drummond and C. Boutilier. Preference elicitation and interview minimization in stable matchings. In *Proceedings of the Twenty-Eighth AAAI Conference on Artificial Intelligence*, pages 645–653, 2014.
3. L. Ehlers and J. Massó. Matching markets under (in)complete information. *Journal of Economic Theory*, 157:295–314, 2015.
4. D. Gale and L. S. Shapley. College admissions and the stability of marriage. *The American Mathematical Monthly*, 69(1):9–15, 1962.
5. D. Gusfield and R. Irving. *The Stable Marriage Problem: Structure and Algorithms*. MIT Press, 1989.
6. D. Gusfield and R. W. Irving. *The stable marriage problem: Structure and algorithms*. MIT Press, Cambridge, MA, USA, 1989.

7. M. M. Halldórsson, K. Iwama, S. Miyazaki, and H. Yanagisawa. Improved approximation results for the stable marriage problem. *ACM Trans. Algorithms*, 3(3), 2007.
8. N. Hazan, Y. Aumann, S. Kraus, and M. Wooldridge. On the evaluation of election outcomes under uncertainty. *Artificial Intelligence*, 189:1–18, 2012.
9. R. Irving. Stable marriage and indifference. *Discrete Applied Mathematics*, 48: 261–272, 1994.
10. Y. Li and V. Conitzer. Cooperative game solution concepts that maximize stability under noise. In *AAAI*, pages 979–985, 2015.
11. D. Manlove. *Algorithmics of Matching Under Preferences*. World Scientific Publishing Company, 2013.
12. D. F. Manlove. Stable marriage with ties and unacceptable partners. Technical Report TR-1999-29, University of Glasgow, Department of Computing Science, 1999.
13. B. Rastegari, A. Condon, N. Immorlica, R. Irving, and K. Leyton-Brown. Reasoning about optimal stable matching under partial information. In *Proceedings of the ACM Conference on Electronic Commerce (EC)*, pages 431–448. ACM, 2014.
14. A. E. Roth and M. A. O. Sotomayor. *Two-Sided Matching: A Study in Game Theoretic Modelling and Analysis*. Cambridge University Press, 1990.
15. V. G. Vizing. On an estimate of the chromatic class of a p-graph. *Diskret. Analiz*, 3(7):25–30, 1964.
16. D. J. A. Welsh and C. Merino. The Potts model and the Tutte polynomial. *Journal of Mathematical Physics*, 41(3):1127–1152, 2000.